BASIC FINANCIAL MATHEMATICS

COMPOUND VALUE

in N periods time of $A at an interest rate of r per cent per period is

F = $A (1 + r)^N$

Example: r = .10 (= 10 %), N = 2, A = $100

F = $100 (1.10)^2$ = $ 121

PRESENT VALUE

of $F due in N period's time at an interest (discount) rate of r per period is

A = $F (1 + r)^{-N}$

A is the amount which if invested today at a rate of interest r which compounds each period will yield F in N period's time.

Example: F = $140, r = .12, N = 3

A = $140 (1.12)^{-3}$ = $99.6492

PRESENT VALUE OF A SEQUENCE

of $F_i due in i period's time where i =1,2,...,I, is

A = $F_1(1+r)^{-1} + F_2(1+r)^{-2} + ... + F_I(1+r)^{-I}$

i.e. the P.V. of a sum is the sum of the P.V.s.

Example: I = 3, F_1 = $40, F_2 = $50, F_3 = $60, r = .10

A = $40 (1.10)^{-1} + 50(1.10)^{-2} + 60(1.10)^{-3}$

= 36.3636 + 41.3223 + 45.0789

= 122.7648

INTERNAL RATE OF RETURN (I.R.R.)

If Y is the market value of the stream F_1, F2, F_3,....F_I,

the discount (interest) rate r which makes Y the Present Value of that stream is known as the internal rate of return.

The I.R.R. is that interest rate at which investment of Y could yield the income stream F_1,...F_I.

N.B. it may not be possible to invest Y in this fashion (easily) since the interest rates available for different periods may differ.

Example: $Y = 100$, $F_1 = 60$, $F_2 = 60$.

The I.R.R. (r) is the solution to

$100 \quad = \quad 60.(1+r)^{-1} + 60.(1+r)^{-2}$

i.e.

$100.r^2 + 140.r - 20 = 0$

$r \quad = \quad .13065 \quad \text{or} \quad -1.53065$

$r \quad = \quad 13.065\ \%$

PRESENT VALUE OF AN ANNUITY

An N period annuity is a stream of N payments, all of the same dollar amount (denoted by X) starting in one period's time.

The P.V. of an annuity is found as follows.

Let k^i represent $(1+r)^{-i}$ i = 1,2,....

$V \quad = Xk^1 + Xk^2 + xk^3 \ldots\ldots \quad \ldots\ldots +Xk^N$

$Vk^1 \quad = \quad Xk^2 + Xk^3 + \ldots \quad \ldots +Xk^N +Xk^{N+1}$

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$V - Vk^1 \quad = \quad Xk^1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots -Xk^{N+1}$

$V \quad = \quad X (k^1 - k^{N+1}) /(1- k^1) = X(1-k^N)k^1/(1-k^1)$

Since $k^1/(1-k^1) = (1+r)^{-1}/(1-(1+r)^{-1}) = 1/r$

$V \quad = \quad X \{ 1 - (1+r)^{-N} \} / r$

Example:

The prize in a competition is $1000 per year for 20 years (starting a year hence) or a one-off payment now of $10000. The interest rate at which you can invest or borrow funds is 12%. What should you do?

Answer: The annual payments are an annuity and its P.V. is

$V \quad = \quad 1000\{1 - 1.12^{-20} \} /.12$

$\quad = \quad \$7478$

You should take the one-off payment.

P.V. OF A PERPETUITY

A perpetuity is an annuity which runs forever.

The P.V. of a perpetuity offering $X per period for ever is given by the limit as N tends to infinity of the annuity formula. This is

$V = X / r$

P.V. OF A PERPETUALLY GROWING STREAM

Suppose the income stream is $X next period and g per cent higher each successive period, i.e. the stream is : X, X(1+g), $X(1+g)^2$..

Let $k^i = (1 + g)^{i-1} / (1 + r)^i$

$V = Xk^1 + Xk^2 + xk^3 \ldots \ldots + Xk^N + \ldots$

$Vk^1 = Xk^2 + Xk^3 + \ldots \ldots + Xk^N + Xk^{N+1} + \ldots$

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$V - Vk^1 = Xk^1$

substituting

$V = Xk^1/(1-k^1) = X / (r - g)$

Example: Suppose a company is expected to pay dividends of: $2.50 next period, and 5 per cent higher each successive year. The interest rate appropriate to the uncertainty associated with that income stream is 15 per cent. What should you pay for that company's shares?

Answer: V = $2.50 /(0.15 - 0.05)

V = $25.00

MORTGAGE REPAYMENTS

A standard mortgage involves regular periodic payments of a constant amount (provided that the interest rate charged by the mortgagee does not change). Each payment involves a (changing) mix of interest and principal repayment.

What is the repayment required (R) if the principal amount borrowed P and interest at rate r% per period is to be repaid by N equal payments?

The answer is obtained by noting that from the mortgagee's perspective, the repayment stream is an annuity. The P.V. of that annuity must be equal to the principal handed over.

Hence:

$P = X\{1 - (1+r)^{-N}\} / r$

$X = r.P / \{1 - (1 + r)^{-N}\}$

Example: Principal $100,000, 25 years, monthly instalments, monthly interest rate 1.5 %.

X = .015 (100,000)/(1 - .0114865)

= 1,500 / .9885

=$1,517.45

N.B. for N large, $X \simeq r.P$, so that

$dX/X \simeq dr/r$

i.e. an increase in r from 10% to 11% increases repayments by almost 10%.

EFFECTIVE INTEREST RATES

The compound (future) value of $1 in I periods is

$C.V.(I) = (1 + r^1)(1 + r^2).....(1 + r^I)$

where r^i is the interest rate for that period (not necessarily p.a.)

If an annual interest rate r is credited at intervals of less than a year, say m times, the interest rate for each of the m periods is r/m. The compound value of $1 at the end of the year will be

$C.V.(1) = (1 + r/m)^m$

There is also some annual interest rate r_e which if credited only once at the end of the year would give the same end of year value C.V.(1), calculated now as:

$C.V.(1) = (1 + r_e)$.

Equating and solving for the <u>effective rate of interest</u> r_e

$r_e = (1 + r/m)^m - 1$

Example: A Semi Government authority offers two types of one year maturity bonds.

Bond A: annual interest rate 12.4 % paid at the end of the year.

Bond B: annual interest rate 12.0 % paid quarterly

(You believe you can reinvest the quarterly interest cheques at 12 %). Which should you purchase?

Answer: $100 invested in A gives $112.40 at the end of the year.

$100 invested in B gives $100 (1.03)^4 =$100 (1.1255) =$112.55

Bond B gives the better return.

CONTINUOUS COMPOUNDING

As compounding becomes more frequent (m increases) the compound value of $1 at year's end approaches $\lim (1 + r/m)^m = e^r$ where e is the exponential number (= 2.71828..)

The effective interest rate with continuous compounding is thus

$r_e \quad = \quad e^r - 1.$

Example:

Is $1 invested for one year at 10 % continuously compounded better or worse than $1 invested at 10.4% with annual compounding?

Answer:

With continuous compounding the end of year value is

$\$1.\ e^{0.10} = \quad \$1.10517$

i.e. an effective annual interest rate of 10.517 %

VALUING GOVERNMENT SECURITIES

The <u>yield to maturity</u> is the I.R.R. associated with holding the security to maturity.

Interest is paid half yearly, at an annual coupon rate of G%, i.e. the half yearly payment per $100 face value will be

G/200 ($ 100).

e.g. if G = 12.64, the half yearly payment is $6.32

<u>If</u> there is exactly a half year to go to the next interest date and there are N interest dates (including the maturity date) - so that the security has a maturity in years of N/2, the I.R.R. (r) can be calculated as follows - where P is the current market price (per $100 face value)

$P \quad = \quad G/2\{ v + v^2 + v^3 + ... + v^N \} + 100 \cdot v^N$

where $v = 1 / (1 + r/2) = 1 / (1 + i)$. [where i is the half yearly rate]

Example:

Coupon rate G = 10%, 2 years to maturity (N = 4), market price P = $ 80.00

$80 \quad = \quad 5 \{ 1 - (1+i)^{-4} \}/i + 100\ (1 + i)^{-4}$

The relevant solution to this equation is i = .11518, giving an annual rate of 23.036 %